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Some Chebyshev Approximations by Polynomials in Two Variables

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1. INTRODUCTION

In this paper we present several examples of functions of two variables for which it is possible to obtain explicit expressions for the Chebyshev approximations by polynomials. The construction of explicit best approximations for functions of more than one variable has been already accomplished in a few cases (see, e.g., [7, 9, 10]). A classical problem in Chebyshev approximation is the determination of the best approximation to x^n on $[-1, 1]$ by a polynomial of degree $n - 1$. Many of the problems we consider are of this type. Specifically, we obtain Chebyshev approximations to a large variety of homogeneous polynomials of degree n on the unit disk by polynomials of degree $n - 1$. We consider also the approximation of certain other functions on the disk.

A general setting for the problem of Chebyshev approximation is the following. Let $\mathscr{D}$ be a compact Hausdorff space, and let $C(\mathscr{D})$ denote the space of continuous real-valued functions on $\mathscr{D}$. Given $f \in C(\mathscr{D})$, its norm is $\|f\| = \max\{|f(x)| : x \in \mathscr{D}\}$. Let V be an n-dimensional subspace of $C(\mathscr{D})$. Given $f \in C(\mathscr{D})$, the problem of Chebyshev approximation is to find $v_* \in V$ such that

$$\|f - v_*\| = \inf\{\|f - v\| : v \in V\}.$$

A signature σ on $\mathscr{D}$ is a function with finite support, whose nonzero values are either $+1$ or -1. We say that σ is extremal with respect to V if there exists a nonzero positive measure μ with carrier in the support of σ such that

$$\int v(x)\, \sigma(x)\, d\mu(x) = 0$$

for all $v \in V$.

For each $v \in V$, we call $f - v$ an error function, and we refer to the set

$$E_f(v) = \{x \in \mathscr{D} : |f(x) - v(x)| = \|f - v\|\}$$

as the extreme points of $f - v$. Now the characterization of the best approximations of f out of V is given in the following result of Rivlin and Shapiro [8].

THEOREM 1.1. *Let* $d = \inf\{\|f - v\| : v \in V\}$. *Then* $v_* \in V$ *satisfies* $\|f - v_*\| = d$ *if and only if there exists an extremal signature* σ *with support in* $E_f(v_*)$ *such that* $(f - v_*)\sigma \geq 0$.

This theorem underlies the work in this paper. Some results concerning the nature of extremal signatures for various subspaces V are presented in [2, 4, 5, 7, 9, 11].

We will take $\mathscr{D}$ to be the unit disk in the plane. Thus,

$$\mathscr{D} = \{(x, y) : x^2 + y^2 \leq 1\},$$

and we shall denote by $\partial\mathscr{D}$ the set of all (x, y) such that $x^2 + y^2 = 1$. By $P_n{}^k$ we designate the space of real polynomials of degree n in k variables. In particular, $p \in P_n{}^2$ has the form

$$p(x, y) = \sum c_{ks} x^k y^s$$

where $k + s \leq n$ and the c_{ks} are real numbers.

There is a particular extremal signature which occurs quite often in this paper. Let $\varphi_1 < \varphi_2 < \cdots < \varphi_{2n}$ be angles in $[0, 2\pi)$ and let $r > 0$ be given. Define the signature σ on the plane by $\sigma(r \cos \varphi_i, r \sin \varphi_i) = (-1)^i$ and $\sigma = 0$ otherwise. It is well known [5, 9] that σ is extremal with respect to P^2_{n-1}. We will refer to this kind of extremal signature as an "alternant of type $2n$."

2. APPROXIMATION OF $x^n y^m$

Let n and m be arbitrary nonnegative integers with $n + m \geq 1$. In this section, we consider the Chebyshev approximation of the function $x^n y^m$ on the unit disk $\mathscr{D}$ by polynomials in P^2_{n+m-1}.

Let $U_k(x)$ denote the polynomial of degree $k \geq 0$ defined by

$$U_k(\cos \varphi) = \sin(k + 1)\varphi/\sin \varphi.$$

Thus, U_k is the Chebyshev polynomial of degree k of the second kind (see [3]). For convenience, we define $U_{-1}(x) = 0$ and $U_{-2}(x) = -1$.

THEOREM 2.1. *For integers $n \geqslant 0$ and $m \geqslant 0$ with $n + m \geqslant 1$, set*

$$P_{n,m}(x, y) = (1/2)^{n+m}(U_n(x)\, U_m(y) + U_{n-2}(x)\, U_{m-2}(y))$$

Then $P_{n,m}$ is an error function of best Chebyshev approximation to $x^n y^m$ on $\mathscr{D}$ out of P^2_{n+m-1}. The deviation of the best approximation is $(1/2)^{n+m-1}$.

Proof. We observe that

$$P_{n,m} = x^n y^m + \{\text{lower degree terms}\}$$

and therefore $P_{n,m}$ has the form of an error function. Let us first consider $P_{n,m}$ restricted to $\partial\mathscr{D}$. Using the trigonometric definition of the polynomials U_k, it is not difficult to show that

$$P_{n,m}(\cos\varphi, \sin\varphi) = \begin{cases} K_{n,m} \sin(n+m)\,\varphi, & m \text{ odd}, \\ K_{n,m} \cos(n+m)\,\varphi, & m \text{ even}, \end{cases} \tag{2.1}$$

where $K_{n,m} = (-1)^{[m/2]}(1/2)^{n+m-1}$ ($[t]$ denotes the largest integer $\leqslant t$).

To show that $P_{n,m}$ is an error function of best approximation, it suffices to show that

$$|P_{n,m}(x, y)| \leqslant (1/2)^{n+m-1}$$

for all $(x, y) \in \mathscr{D}$. For then there would be an alternate σ of type $2(n+m)$ on $\partial\mathscr{D}$, with support in the extreme points of $P_{n,m}$, such that $\sigma P_{n,m} \geqslant 0$ on $\mathscr{D}$.

Set $x = \cos\varphi$ and $y = \cos\theta$. Then

$$P_{n,m} = (1/2)^{n+m-1} F_{n,m}(\varphi, \theta)$$

where

$$F_{n,m}(\varphi, \theta) = \left(\frac{1}{2}\right) \frac{\sin(n+1)\,\varphi \sin(m+1)\,\theta + \sin(n-1)\,\varphi \sin(m-1)\,\theta}{\sin\varphi \sin\theta}.$$

We need to show that $|F_{n,m}(\varphi, \theta)| \leqslant 1$ for all (φ, θ) such that $\cos^2\varphi + \cos^2\theta \leqslant 1$. If either $\varphi = 0$ or $\theta = 0$, then $(x, y) \in \partial\mathscr{D}$, and here we already know that $|F_{n,m}| \leqslant 1$. Hence, we need consider only φ and θ in $(0, \pi)$. Applying the identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, to the numerator of $F_{n,m}$, we obtain

$$F_{n,m} = A \sin n\varphi \sin m\theta + \cos n\varphi \cos m\theta$$

where

$$A = (\cos\varphi/\sin\theta) \cdot (\cos\theta/\sin\varphi).$$

Since φ and θ belong to $(0, \pi)$ and $\cos^2\varphi + \cos^2\theta \leqslant 1$, it follows that $|\cos\varphi/\sin\theta| \leqslant 1$ and $|\cos\theta/\sin\varphi| \leqslant 1$. Hence $|A| \leqslant 1$, and therefore

$$|F_{n,m}| \leqslant |\sin n\varphi|\,|\sin m\theta| + |\cos n\varphi|\,|\cos m\theta|.$$

Using the Schwarz inequality, we see that $|F_{n,m}| \leqslant 1$, as was to be shown.

From this theorem, it is possible to construct certain other best approximations to $x^n y^m$. Let $T_k(x)$ be the Chebyshev polynomial of the first kind of degree k [3]. Thus, $T_k(\cos \varphi) = \cos k\varphi$ for $k = 0, 1, 2, \ldots$. Let $p_{n,m}$ denote any error function of best approximation to $x^n y^m$ out of P^2_{n+m-1} on $\mathscr{D}$. Then

$$-1 \leqslant 2^{n+m-1} p_{n,m}(x, y) \leqslant 1$$

for all $(x, y) \in \mathscr{D}$. Now consider the polynomial

$$P(x, y) = 2^{-k(n+m)+1} T_k[2^{n+m-1} p_{n,m}(x, y)]. \tag{2.2}$$

It is easy to verify that P is a polynomial of the form $x^{kn} y^{km}$ + (lower degree terms), and that furthermore $|P(x, y)| \leqslant 2^{-k(n+m)+1}$ for all $(x, y) \in \mathscr{D}$. Thus, by considering Theorem 2.1 we obtain

COROLLARY 2.1. *If $T_k(x)$ is the Chebyshev polynomial of the first kind, and $p_{n,m}$ is any best error function for $x^n y^m$ on $\mathscr{D}$, then the polynomial* (2.2) *is a best error function for $x^{kn} y^{km}$ on $\mathscr{D}$.*

In general, it can be seen that this polynomial (2.2) is distinct from the $P_{kn,km}$ of Theorem 2.1. We suspect therefore that the best Chebyshev approximations to $x^n y^m$ on $\mathscr{D}$ are not unique. A complete answer to the uniqueness question is contained in the following theorem.

THEOREM 2.2. *The best approximation to $x^n y^m$ on $\mathscr{D}$ out of P^2_{n+m-1} is unique only if $n = 0$ or $m = 0$, or $n = m = 1$. For the case $n \geqslant 1$, $m \geqslant 1$ and $n + m \geqslant 3$, we have:*

(a) *If $P(x, y)$ is an error function of best approximation to $x^n y^m$, then*

$$P = P_{n,m} + (1 - x^2 - y^2) Q$$

for some $Q \in P^2_{n+m-3}$.

(b) *There is a polynomial $Q_* \in P^2_{n+m-3}$, which is not identically zero, such that*

$$P_* = P_{n,m} + (1 - x^2 - y^2)\, Q_*$$

is an error function of best approximation to $x^n y^m$ on $\mathscr{D}$ out of P^2_{n+m-1}. Moreover, Q_* has the following property: Given any $Q \in P^2_{n+m-3}$, there corresponds a constant $\Lambda = \Lambda(Q) > 0$ such that for any λ with $0 \leqslant \lambda \leqslant \Lambda$,

$$P_* + \lambda(1 - x^2 - y^2) Q$$

is an error function of best approximation to $x^n y^m$ on $\mathscr{D}$ out of P^2_{n+m-1}.

The proof of this and related results can be found in [6]. We omit the proof here as it is somewhat long and merely technical in nature. We observe that Theorem 2.2 may be viewed as a characterization of the best approximations to $x^n y^m$ on $\mathscr{D}$ out of P^2_{n+m-1}. The polynomial Q_* in (b) is not unique, and need satisfy only mild restrictions. In fact, as shown in [6], it is not difficult to construct many such Q_* explicitly.

The uniqueness result for $n = 0$ or $m = 0$ can be obtained in a more general setting.

THEOREM 2.3. *Suppose $f = f(y)$ is a continuous function on $\mathscr{D}$. Let $p_*(y)$ be the polynomial of degree k which best approximates $f(y)$ on* $[-1, 1]$. *Then p_* is the unique best approximation of f out of P_k^2 on $\mathscr{D}$.*

Proof. If $k = 0$, the result is obvious and so we assume $k \geqslant 1$. It is clear that $p_* \in P_k^2$ is a best approximation to f on $\mathscr{D}$. Suppose $p \in P_k^2$ is such that $p_* + p$ is another best approximation to f on $\mathscr{D}$. Since p_* is characterized by the alternation theorem [3], we can find at least k lines $y = y_i$, $i = 1, 2,..., k$ where $-1 < y_1 < y_2 < \cdots < y_k < 1$ such that each line is contained in the extremal points of $f - p_*$ on $\mathscr{D}$, and $f - p_*$ alternates in sign on these lines. Now, p_* is the only best approximation to f on $[-1, 1]$, and therefore $p(0, y) = 0$ for all y. Hence, $p(x, y) = xP(x, y)$ for some $P \in P^2_{k-1}$. Select $h > 0$ such that $[-h, h] \times [y_1, y_k]$ is contained in the interior of $\mathscr{D}$. For definiteness, let us assume that the line $y = y_1$ is a negative line. Then, since $p_* + p$ is a best approximation, it follows that for each $x \in [-h, h]$, $(-1)^i p(x, y_i) \leqslant 0$, $i = 1, 2,..., k$. Thus, for each i, $(-1)^i P(x, y_i) \leqslant 0$ if $x > 0$ and $(-1)^i P(x, y_i) \geqslant 0$ if $x < 0$, and therefore, $P(0, y_i) = 0$ for $i = 1, 2,..., k$. But $P(0, \cdot)$ is a polynomial of degree at most $k - 1$. In particular, if $k = 1$, then we must conclude that $P(x, y) = 0$ for all (x, y) and thus the result is valid when $k = 1$. For $k \geqslant 2$, however, it follows that $P(x, y) = xQ(x, y)$ for some $Q \in P^2_{k-2}$, and thus $p(x, y) = x^2Q(x, y)$. Now let x_0 be such that $-h \leqslant x_0 \leqslant h$. Then $(-1)^i Q(x_0, y_i) \leqslant 0$ for $i = 1, 2,..., k$. It follows that the polynomial $Q(x_0, \cdot)$ has at least $k - 1$ zeros, counting multiplicities. Hence, $Q(x_0, y) = 0$ for all y. But x_0 was an arbitrary point in $[-h, h]$, and therefore $Q(x, y) = 0$ for all (x, y). The proof is thus complete.

We remark that the proof of this result did not require that $\mathscr{D}$ be a disk. In fact, we could have taken $\mathscr{D}$ to be any compact set lying between the lines $y = 1$ and $y = -1$ and whose interior contains the interval $(-1, 1)$ on the y-axis.

3. Approximations Obtained from $P_{n,m}$

A real homogeneous polynomial of degree $n \geqslant 1$ has the form

$$p_n(x, y) = \sum_{k=0}^{n} c_k x^{n-k} y^k \tag{3.1}$$

Using the best approximations of $x^{n-k}y^k$ for $k = 0, 1, 2,\ldots, n$, it is possible to construct best approximations to a variety of homogeneous polynomials on $\mathscr{D}$. We state first

THEOREM 3.1. *Suppose the c_k in (3.1) satisfy one of the following two conditions*:

(a) $c_{2s}c_{2s+2} \leqslant 0$ *and* $c_{2s+1} = 0$ *all* $s = 0, 1, 2,\ldots$.
(b) $c_{2s+1}c_{2s+3} \leqslant 0$ *and* $c_{2s} = 0$ *all* $s = 0, 1, 2,\ldots$.

Let $P_{n-k,k}$ for $k = 0, 1, 2,\ldots, n$ denote any error function of best approximation to $x^{n-k}y^k$ on $\mathscr{D}$. Then $\sum_{k=0}^{n} c_k P_{n-k,k}$ is an error function of best Chebyshev approximation to p_n (3.1) on $\mathscr{D}$ by polynomials in P^2_{n-1}.

Proof. We will show only case (a), as the proof for case (b) is similar. Set

$$P_n = \sum_{k=0}^{n} c_k P_{n-k,k} .$$

From property (a) and (2.1),

$$\begin{aligned} P_n(\cos\varphi, \sin\varphi) &= \sum_{k \text{ even}} c_k K_{n-k,k} \cos n\varphi \\ &= A_n \cos n\varphi \end{aligned}$$

where

$$A_n = (1/2)^{n-1} \sum_{k \text{ even}} c_k(-1)^{[k/2]}.$$

Hence

$$\begin{aligned} |A_n| &= (1/2)^{n-1} \sum_{k \text{ even}} |c_k| \\ &= (1/2)^{n-1} \sum_{k=0}^{n} |c_k|. \end{aligned}$$

But, we recall that $|P_{n-k,k}| \leqslant (1/2)^{n-1}$ on $\mathscr{D}$, and therefore

$$|P_n(x, y)| \leqslant (1/2)^{n-1} \sum_{k=0}^{n} |c_k| = |A_n|$$

for all $(x, y) \in \mathscr{D}$. We see now that there is an alternant σ of type $2n$ on $\partial\mathscr{D}$, with support in the extreme points of P_n, such that $\sigma P_n \geqslant 0$ on $\mathscr{D}$. Hence, P_n is an error function of best approximation.

From this theorem, we obtain best approximations to a large variety of homogeneous polynomials on $\mathscr{D}$. Nevertheless, the conditions (a) and (b) do not include all cases in which $\sum_{k=0}^{n} c_k P_{n-k,k}$ is an error function of best approximation for p_n. For example, consider the polynomial

$$p_n = ax^n + bx^{n-1}y \tag{3.2}$$

where a and b are arbitrary real numbers. Then an error function of best approximation by polynomials in P_{n-1}^2 on $\mathscr{D}$ is given by $Q_n = aP_{n,0} + bP_{n-1,1}$, where $P_{n,0}(x, y) = (1/2)^{n-1} T_n(x)$ and $P_{n-1,1}(x, y) = (1/2)^{n-1} yU_{n-1}(x)$.

Proof. On $\partial\mathscr{D}$, we have

$$Q_n(\cos\varphi, \sin\varphi) = (1/2)^{n-1}(a^2 + b^2)^{1/2} \sin(n\varphi + \alpha)$$

for some angle α. Now, suppose (x_0, y_0) is a point in the interior of $\mathscr{D}$ at which Q_n attains its largest magnitude on $\mathscr{D}$. Then

$$\partial Q(x_0, y_0)/\partial y = (1/2)^{n-1} bU_{n-1}(x_0) = 0.$$

Thus, $bP_{n,1}(x_0, y_0) = 0$, and therefore,

$$|Q_n(x, y)| \leqslant (1/2)^{n-1} |aT_n(x_0)| \leqslant (1/2)^{n-1} |a| \leqslant (1/2)^{n-1}(a^2 + b^2)^{1/2}$$

for all $(x, y) \in \mathscr{D}$. Hence, there is an alternant σ of type $2n$, with support in the extreme points of Q_n such that $\sigma Q_n \geqslant 0$.

If a and b are nonzero, then p_n (3.2) does not satisfy the conditions of Theorem 3.1, but the best error function Q_n is of the form $\sum_{k=0}^{n} c_k P_{n-k,k}$.

We consider now the Chebyshev approximation on $\mathscr{D}$ of $x^n(x^2 + y^2)^m$ by polynomials in P_{n+2m-1}^2. The alternants of type $2k$ which arose in all the previous problems do not arise here.

THEOREM 3.2. *Let $P_{n,2m}(x, y)$ be an error function of best approximation to $x^n y^{2m}$ on $\mathscr{D}$, in which all powers of y are even. Then*

$$Q_{n,m}(x, y) = (-1)^m P_{n,2m}(x, (1 - x^2 - y^2)^{1/2})$$

is an error function of best Chebyshev approximation to $x^n(x^2 + y^2)^m$ on $\mathscr{D}$ out of P_{n+2m-1}^2.

Proof. It is clear that $Q_{n,m}$ has the form $x^n(x^2 + y^2)^m$ + {lower degree terms}, and since $x^2 + ((1 - x^2 - y^2)^{1/2})^2 \leqslant 1$ whenever $(x, y) \in \mathscr{D}$, we have

$$| Q_{n,m}(x, y)| \leqslant (1/2)^{n+2m-1}$$

for all $(x, y) \in \mathscr{D}$. But on the line $y = 0$ in $\mathscr{D}$, we have

$$Q_{n,m}(x, 0) = (-1)^m P_{n,2m}(x, (1 - x^2)^{1/2}) = (1/2)^{n+2m-1} T_{n+2m}(x).$$

Thus, the deviation of $Q_{n,m}$ on $\mathscr{D}$ is $(1/2)^{n+2m-1}$ and on the line $y = 0$ in $\mathscr{D}$, this deviation is least possible. Hence $Q_{n,m}$ is an error function of best approximation.

Using the best approximations to $x^n(x^2 + y^2)^m$, we can obtain best approximations to certain homogeneous polynomials of the form

$$P_n(x, y) = \sum_{k=0}^{[n/2]} c_k x^{n-2k}(x^2 + y^2)^k. \tag{3.2}$$

The result here is similar to that of Theorem 3.1.

THEOREM 3.3. *Let $Q_{n,m}(x, y)$ denote any error function of best approximation to $x^n(x^2 + y^2)^m$ on $\mathscr{D}$ out of P^2_{n+2m-1}. Suppose all the c_k in* (3.2) *have the same sign. Then $\sum_{k=0}^{[n/2]} c_k Q_{n-2k,k}$ is a best error function for P_n* (3.2) *on $\mathscr{D}$ out of P^2_{n-1}.*

Proof. $Q_{n,m}(x, 0)$ is a polynomial of degree $n + 2m$, and the coefficient of the x^{n+2m} term is one. Since the deviation of $Q_{n,m}(x, 0)$ is $(1/2)^{n+2m-1}$ for $-1 \leqslant x \leqslant 1$, it follows that

$$Q_{n,m}(x, 0) = (1/2)^{n+2m-1} T_{n+2m}(x).$$

Hence

$$P_n(x, 0) = (1/2)^{n+2m-1} \sum_{k=0}^{[n/2]} c_k T_n(x) = A_n T_n(x), \tag{3.3}$$

where

$$A_n = (1/2)^{n+2m-1} \sum_{k=0}^{[n/2]} c_k .$$

Since all the c_k have the same sign,

$$| A_n | = (1/2)^{n+2m-1} \sum_{k=0}^{[n/2]} | c_k |.$$

But, for any $(x, y) \in \mathscr{D}$,

$$| P_n | \leqslant \sum_{k=0}^{[n/2]} | c_k | | Q_{n-2k,k} | \leqslant | A_n |.$$

Hence, the deviation of P_n on $\mathscr{D}$ is $| A_n |$, and because of (3.3), this deviatio is least possible.

4. Other Explicit Approximations

Suppose f is a continuous real-valued function on [0, 1] such that $f(0) = ($ Let $m \geqslant 1$ be an integer, and define the function $F \in C(\mathscr{D})$ by

$$F(x, y) = f(r) \cos m\varphi \tag{4.1}$$

where $x = r \cos \varphi$ and $y = r \sin \varphi$.

THEOREM 4.1. *Let $F \in C(\mathscr{D})$ be given by* (4.1), *and let V_k denote the spac of all polynomials of the form $x^m p(x^2)$ where $p \in P_k^1$. Then a best Chebyshe approximation to F on $\mathscr{D}$ out of P_n^2 is*

(a) 0 *if* $n \leqslant m - 1$

(b) $p_*(r^2)\, r^m \cos m\varphi$, *if* $n \geqslant m$, *where* $r^m p_*(r^2)$ *is a best approximatio to* f *on* [0, 1] *out of* $V_{[(n-m)/2]}$.

Proof. (*a*) Let $r_0 \in (0, 1]$ be such that

$$| f(r_0) | = \max_{[0,1]} | f(r) |.$$

Then, on the circle of radius r_0, there is an alternant σ of type $2m$, havin support in the extreme points of F, such that $\sigma F \geqslant 0$ on $\mathscr{D}$. Hence, th polynomial 0 is a best approximation if $n \leqslant m - 1$.

(*b*) For $n \geqslant m$, let $r^m p_*(r^2)$ be a best approximation to f on [0, 1] ou of $V_{[(n-m)/2]}$, and let σ be an associated primitive extremal signature. Sinc $f(0) = 0$, the support of σ consists of $N = [(n - m)/2] + 2$ points r_i i $(0, 1]$.

Consider now the error function

$$E = F - r^m p_*(r^2) \cos m\varphi.$$

Let $\alpha(\varphi)$ be given by $\alpha(\pi i/m) = (-1)^i$ for $i = 0, 1, 2, \ldots, 2m - 1$, wit $\alpha = 0$ elsewhere. Define the signature μ on the plane by $\mu(x, y) = \sigma(r)\alpha(\varphi)$ where $x = r \cos \varphi$ and $y = r \sin \varphi$. We see that the support of μ is containe in the extreme points of E, and that $\mu E \geq 0$ on $\mathscr{D}$. Hence, it suffices to show that μ is extremal with respect to P_n^2.

This result can be shown using the method of Shapiro [9]. Indeed, define the polynomials

$$P_1(x, y) = \prod_{i=1}^{N} (x^2 + y^2 - r_i^2)$$

$$P_2(x, y) = \prod_{i=0}^{m-1} (a_i x + b_i y)$$

where $a_i x + b_i y = 0$ is the equation of the line which passes through the origin and the point $(\cos \pi i/m, \sin \pi i/m)$. Then the support of μ coincides with the set of common roots of P_1 and P_2. In the notation of [9], $m_1 = 2N$ and $m_2 = m$, and by checking the sign of the Jacobian at the common roots, we find that μ is an extremal signature for polynomials of degree

$$\begin{aligned} m_1 + m_2 - 3 &= 2([(n-m)/2] + 2) + m - 3 \\ &\geq (n - m - 1 + 4) + m - 3 \\ &\geq n. \end{aligned}$$

From this result, we obtain a best approximation to F on $\mathscr{D}$ out of P_n^2 by constructing the best approximation to f on [0, 1] out of $V_{[(n-m)/2]}$. This latter problem can be solved numerically using the Second Remes Algorithm (see [3, p. 99]). The proof of this statement is a trivial modification of the convergence proof in [3].

We consider now one further approximation problem in which we take advantage of the rotation invariance of the disk. Let $u_k(x, y)$ designate a homogeneous, harmonic polynomial of degree $k \geq 1$, and let λ be a fixed but arbitrary real number. We investigate the Chebyshev approximation of the polynomial

$$P_{2n} = u_{2n}(x, y) + \lambda(x^2 + y^2), \qquad n \geq 1$$

by polynomials in P_{2n-1}^2. Without loss of generality, we may assume that

$$u_{2n}(r \cos \varphi, r \sin \varphi) = r^{2n} \cos 2n\varphi.$$

Let us introduce the set X of points (x, y) such that $0 \leq x \leq 1$ and $-\pi \leq y \leq \pi$. We will view P_k^1 as a subspace of the space of continuous functions on X, with a polynomial in P_k^1 considered as a function of the x-variable.

LEMMA 4.1. *If p_* is a best Chebyshev approximation to $x^n \cos y + \lambda x^n$ on X out of P^1_{n-1}, then $p_*(x^2 + y^2)$ is a best approximation to P_{2n} on $\mathscr{D}$ out of P^2_{2n-1}.*

Proof. Since P_{2n} is invariant under a rotation through $2\pi/2n$, there exists a best approximation in P^2_{2n-1} which is also invariant under a rotation through $2\pi/2n$. But a polynomial in P^2_{2n-1} with this property must be of the form $p(x^2 + y^2)$ for some $p \in P^1_{n-1}$. However, for any $p \in P^1_{n-1}$, we have

$$\max_{\substack{r\in[0,1]\\ \varphi\in[-\pi,\pi]}} |r^{2n} \cos 2n\varphi + \lambda r^{2n} + p(r^2)| = \max_{\substack{x\in[0,1]\\ y\in[-\pi,\pi]}} |x^n \cos y + \lambda x^n + p(x)|.$$

Hence, the desired conclusion follows.

The problem, therefore, is reduced to that of finding a best approximation to $x^n \cos y + \lambda x^n$ on X out of P^1_{n-1}. It is not difficult to characterize the extremal signatures on X with respect to P^1_{n-1}. Indeed, any such signature σ must be one of the following two types.

(1) The support of σ consists of two points (x, y_1) and (x, y_2) with $\sigma(x, y_1) = -\sigma(x, y_2)$. We shall call this signature an "opposite sign" extremal signature.

(2) The support of σ consists of $n + 1$ points (x_i, y_i) where $x_1 < x_2 \cdots < x_{n+1}$ and $\sigma(x_i, y_i) = -\sigma(x_{i+1}, y_{i+1})$ for $i = 1, 2,..., n$. We will refer to this signature as an "alternating" extremal signature.

It turns out that both the "opposite sign" and the "alternating" extremal signatures occur in this approximation problem, depending on the value of λ.

LEMMA 4.2. *A polynomial $p \in P^1_{n-1}$ is a best approximation to $x^n \cos y + \lambda x^n$ on X with "opposite sign" extremal signature if and only if*

$$-(1 - x^n) \leqslant \lambda x^n - p(x) \leqslant 1 - x^n \tag{4.3}$$

for all $x \in [0, 1]$. Moreover, the deviation of this best approximation on X is one.

Proof. Suppose $p \in P^1_{n-1}$ is a best approximation to $x^n \cos y + \lambda x^n$ on X with "opposite sign" extremal signature. Let (ξ, y_1) and (ξ, y_2) denote the support of this extremal signature, and let $F(x, y) = x^n \cos y + \lambda x^n - p(x)$ be the error function.

Now, it cannot be the case the $\xi = 0$, since $F(0, y) = p(0)$ does not change sign as y varies. But, for all $(x, y) \in X$,

$$-x^n + \lambda x^n - p(x) \leqslant F(x, y) \leqslant x^n + \lambda x^n - p(x), \tag{4.4}$$

and if $x \in (0, 1]$, we have equality on the right only when $y = 0$, and we have equality on the left only when $y = \pm\pi$. Hence, we may assume that $y_1 = 0$, and $y_2 = \pi$.

Now, by hypothesis, $F(\xi, 0) = -F(\xi, \pi)$, and this equation implies that $\lambda\xi^n - p(\xi) = 0$. Hence,

$$\max_{(x,y)\in X} |F(x, y)| = |F(\xi, 0)| = \xi^n, \tag{4.5}$$

and therefore, for all $x \in [0, 1]$

$$F(x, 0) = x^n + \lambda x^n - p(x) \leqslant \xi^n$$

and

$$F(x, \pi) = -x^n + \lambda x^n - p(x) \geqslant \xi^n.$$

Combining these inequalities, we obtain

$$-(\xi^n - x^n) \leqslant \lambda x^n - p(x) \leqslant (\xi^n - x^n)$$

for all $x \in [0, 1]$. In particular, $-(\xi^n - x^n) \leqslant (\xi^n - x^n)$, and substituting $x = 1$ in this inequality yields $\xi = 1$. Thus, (4.3) follows and it is clear from Eq. (4.5) that the deviation of this best approximation $p \in P_{n-1}^1$ is one.

Conversely, suppose that for some $p \in P_{n-1}^1$, inequality (4.3) is satisfied. Then from (4.4), we have for all $(x, y) \in X$,

$$-x^n - (1 - x^n) \leqslant F(x, y) \leqslant x^n + (1 - x^n)$$

so that

$$-1 \leqslant F(x, y) \leqslant 1.$$

But, for the points $(1, 0)$ and $(1, \pi)$ in X,

$$F(1, 0) = 1 + [\lambda - p(1)] = 1$$

and

$$F(1, \pi) = -1 + [\lambda - p(1)] = -1.$$

Thus, $p \in P_{n-1}^1$ is a best approximation with "opposite sign" extremal signature on the points $(1, 0)$ and $(1, \pi)$.

LEMMA 4.3. *Let $B_{n-1}(x) = 1 + x + x^2 \cdots + x^{n-1}$ and define λ_n as follows*: $\lambda_1 = 1$ *and for* $n \geqslant 2$,

$$1/\lambda_n = \inf_{p\in P_{n-2}^1} \sup_{x\in[0,1]} |(x^{n-1} - p(x))/B_{n-1}(x)|.$$

There exists a polynomial in P_{n-1}^1 which satisfies (4.3) *if and only if* $|\lambda| \leqslant \lambda_n$.

Proof. Suppose first that $|\lambda| \leqslant \lambda_n$. If $\lambda = 0$ or $n = 1$, then $p(x) = \lambda$ will satisfy (4.3). Thus we may assume that $\lambda \neq 0$ and that $n \geqslant 2$. Now, $0 < |\lambda| \leqslant \lambda_n$ implies that $1/|\lambda| \geqslant 1/\lambda_n$, and therefore there exists a $q \in P_{n-2}^1$ such that

$$1/|\lambda| \geqslant \sup_{[0,1]} |(x^{n-1} - q(x))/B_{n-1}(x)|$$

Hence

$$-B_{n-1}(x) \leqslant -\lambda(x^{n-1} - q(x)) \leqslant B_{n-1}(x)$$

for all $x \in [0, 1]$. Multiplying this inequality by $1 - x$, we obtain

$$-(1 - x^n) \leqslant -\lambda(1 - x)(x^{n-1} - q(x)) \leqslant 1 - x^n,$$

and the polynomial between the inequality signs has the form required of (4.3).

Conversely, suppose that (4.3) holds for some λ and $p \in P_{n-1}^1$. If $\lambda = 0$, then $|\lambda| \leqslant \lambda_n$, so we will assume that $|\lambda| > 0$. Also, for $n = 1$, it is clear that $|\lambda| \leqslant \lambda_n = 1$; and therefore we will take $n \geqslant 2$. Now, dividing (4.3) by $1 - x$, we obtain

$$-B_{n-1}(x) \leqslant (\lambda x^n - p(x))/(1 - x) \leqslant B_{n-1}(x).$$

But $\lambda x^n - p(x)$ has a root at $x = 1$. Hence

$$\lambda x^n - p(x) = -\lambda(1 - x)(x^{n-1} - q(x)),$$

for some $q \in P_{n-2}^1$. Therefore

$$|(x^{n-1} - q(x))/B_{n-1}(x)| \leqslant 1/|\lambda|$$

so that $1/|\lambda| \geqslant 1/\lambda_n$, and thus $|\lambda| \leqslant \lambda_n$.

We notice that $1/\lambda_n$ is defined as the deviation of the best weighted Chebyshev approximation on [0, 1] of x^{n-1} by polynomials of degree $\leqslant n - 2$, with weight function $B_{n-1}(x)$. Let $p_{n-2} \in P_{n-2}^1$ denote the (unique) polynomial which attains the deviation $1/\lambda_n$. Using a complex variable technique similar to that illustrated in Achieser [1, pp. 278–285], it is possible to obtain explicit expressions for p_{n-2} and λ_n. As it is fairly straightforward to apply the technique in this case, we present only the final result. For $k = 1, 2,..., n - 1$, let $\alpha_k = e^{i(2k\pi/n)}$ and let β_k satisfy

$$\beta_k^2 + 2(1 - 2\alpha_k)\beta_k + 1 = 0, \qquad |\beta_k| < 1.$$

Set

$$f_n(z) = \prod_{k=1}^{n-1} \frac{1 - \bar{\beta}_k z}{z - \beta_k},$$

and let

$$u_n(x) = \frac{x^{n-1} - p_{n-2}(x)}{B_{n-1}(x)}.$$

If we set $x = (1/2)(\cos \varphi + 1)$, with $\varphi \in [0, \pi]$, then

$$u_n(x) = d_n \operatorname{Re}\{f_n(e^{i\varphi})\} \qquad (\mathrm{Re} = \text{real part}) \tag{4.6}$$

where

$$d_n = (-1)^{n-1}\, 2\beta_1\beta_2 \cdots \beta_{n-1}/(1 + (\beta_1 \cdots \beta_{n-1})^2).$$

In particular, for $n \geqslant 2$ we have $\lambda_n = 1/|\, d_n \,|$.

Let us summarize the above results. Define λ_n and $B_{n-1}(x)$ as in Lemma 4.3.

THEOREM 4.2. *Suppose* $|\,\lambda\,| \leqslant \lambda_n$. *Then a polynomial of best approximation to* P_{2n} *on* $\mathscr{D}$ *out of* P^2_{2n-1} *is* $p_*(x^2 + y^2)$, *where* $p_* \in P^1_{n-1}$ *is given by*

(a) λ, *if* $\lambda = 0$ *or* $n = 1$

(b) $\lambda x^n + \lambda(1 - x)(x^{n-1} - q(x))$, *where* $q(x)$ *is any polynomial in* P^1_{n-2} *such that*

$$\max_{[0,1]} |(x^{n-1} - q(x))/B_{n-1}(x)| \leqslant 1/|\,\lambda\,|$$

In particular, this inequality is satisfied by $q(x) = p_{n-2}(x)$, *where* $p_{n-2} \in P^1_{n-2}$ *is determined by* (4.6). *Finally, the deviation of* $P_{2n}(x, y) - p_*(x^2 + y^2)$ *on* $\mathscr{D}$ *is one.*

For the case $|\,\lambda\,| > \lambda_n$, we have shown that the best approximation (on X) is characterized by the "alternating" extremal signature. However, we have not attempted to construct an explicit best approximation.

5. CONCLUSION

There are functions other than those discussed here, for which it is not difficult to obtain explicit best approximations. However, the problem of constructing an explicit best approximation to an arbitrary polynomial of degree n by polynomials of degree $n - 1$ on $\mathscr{D}$ remains unsolved. From the results of Theorem 4.2, it appears that the solution of this problem may be somewhat complicated. We suspect that a large variety of extremal signatures occur in this problem, and that consequently, it is difficult to construct solutions explicitly.

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